AQM1-2
chp1: Translation Operator
chp2: Quantum Dynamics

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گذار از عملگرهای گسسته به عملگرهای پیوسته

$$\xi|\xi'\rangle = \xi'|\xi'\rangle$$

$$\langle a'|a''\rangle = \delta_{a'a''} \rightarrow \quad \langle\xi'|\xi''\rangle = \delta(\xi'-\xi''),$$

$$\sum_{a'} |a'\rangle\langle a'| = 1 \rightarrow \quad \int d\xi' |\xi'\rangle\langle\xi'| = 1,$$

$$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle \rightarrow \quad |\alpha\rangle = \int d\xi' |\xi'\rangle\langle\xi'|\alpha\rangle,$$

$$\sum_{a'} |\langle a'|\alpha\rangle|^2 = 1 \rightarrow \quad \int d\xi' |\langle\xi'|\alpha\rangle|^2 = 1,$$

$$\langle\beta|\alpha\rangle = \sum_{a'} \langle\beta|a'\rangle\langle a'|\alpha\rangle \rightarrow \quad \langle\beta|\alpha\rangle = \int d\xi' \langle\beta|\xi'\rangle\langle\xi'|\alpha\rangle,$$

$$\langle a''|A|a'\rangle = a'\delta_{a'a''} \rightarrow \quad \langle\xi''|\xi|\xi'\rangle = \xi'\delta(\xi''-\xi').$$

ویژه کت های مکان و اندازه گیری مکان

$$x|x'\rangle = x'|x'\rangle$$

$$|\alpha\rangle = \int_{-\infty}^{\infty} dx'|x'\rangle\langle x'|\alpha\rangle$$

$$|\alpha\rangle = \int_{-\infty}^{\infty} dx''|x''\rangle\langle x''|\alpha\rangle \xrightarrow{\text{measurement}} \int_{x'-\Delta/2}^{x'+\Delta/2} dx''|x''\rangle\langle x''|\alpha\rangle$$

$$|\langle x'|\alpha\rangle|^2 dx'$$

$$\int_{-\infty}^{\infty} dx'|\langle x'|\alpha\rangle|^2$$

is normalized to unity if $|\alpha\rangle$ is normalized:

$$\langle\alpha|\alpha\rangle = 1 \Rightarrow \int_{-\infty}^{\infty} dx'\langle\alpha|x'\rangle\langle x'|\alpha\rangle = 1$$

عملگر مکان / مختصات

$$|\alpha\rangle = \int d^3x' |\mathbf{x}'\rangle\langle\mathbf{x}'|\alpha\rangle, \tag{1.6.9}$$

where $\mathbf{x}'$ stands for x', y', and z'; in other words, $|\mathbf{x}'\rangle$ is a *simultaneous* eigenket of the observables x, y, and z in the sense of Section 1.4:

$$|\mathbf{x}'\rangle \equiv |x', y', z'\rangle, \tag{1.6.10a}$$

$$x|\mathbf{x}'\rangle = x'|\mathbf{x}'\rangle, \qquad y|\mathbf{x}'\rangle = y'|\mathbf{x}'\rangle, \qquad z|\mathbf{x}'\rangle = z'|\mathbf{x}'\rangle. \tag{1.6.10b}$$

To be able to consider such a simultaneous eigenket at all, we are implicitly assuming that the three components of the position vector can be measured simultaneously to arbitrary degrees of accuracy; hence, we must have

$$[x_i, x_j] = 0, \tag{1.6.11}$$

where x_1, x_2, and x_3 stand for x, y, and z, respectively.

عملگر انتقال در مکان (فضا)

infinitesimal translation by $d\mathbf{x}'$: $\mathscr{T}(d\mathbf{x}')|\mathbf{x}'\rangle = |\mathbf{x}' + d\mathbf{x}'\rangle$, (1.6.12)

$$|\alpha\rangle \to \mathscr{T}(d\mathbf{x}')|\alpha\rangle = \mathscr{T}(d\mathbf{x}')\int d^3x'|\mathbf{x}'\rangle\langle\mathbf{x}'|\alpha\rangle = \int d^3x'|\mathbf{x}'+d\mathbf{x}'\rangle\langle\mathbf{x}'|\alpha\rangle. \quad (1.6.13)$$

$$\int d^3x'|\mathbf{x}'+d\mathbf{x}'\rangle\langle\mathbf{x}'|\alpha\rangle = \int d^3x'|\mathbf{x}'\rangle\langle\mathbf{x}'-d\mathbf{x}'|\alpha\rangle \quad (1.6.14)$$

We now list the properties of the infinitesimal translation operator

be unitary: $\mathscr{T}^\dagger(d\mathbf{x}')\mathscr{T}(d\mathbf{x}') = 1.$ (1.6.16)

$$\mathscr{T}(d\mathbf{x}'')\mathscr{T}(d\mathbf{x}') = \mathscr{T}(d\mathbf{x}'+d\mathbf{x}''). \quad (1.6.17)$$

$$\mathscr{T}(-d\mathbf{x}') = \mathscr{T}^{-1}(d\mathbf{x}'). \quad (1.6.18)$$

$$\lim_{d\mathbf{x}'\to 0} \mathscr{T}(d\mathbf{x}') = 1 \quad (1.6.19)$$

We now demonstrate that if we take the infinitesimal translation operator to be

$$\mathscr{T}(d\mathbf{x}') = 1 - i\mathbf{K}\cdot d\mathbf{x}', \quad (1.6.20)$$

where the components of $\mathbf{K}$, K_x, K_y, and K_z, are **Hermitian operators**, then all the properties listed are satisfied.

$$[\mathbf{x}, \mathscr{T}(d\mathbf{x}')] = d\mathbf{x}', \qquad (1.6.25)$$

$$[x_i, K_j] = i\delta_{ij}, \qquad (1.6.27)$$

J. Schwinger, lecturing on quantum mechanics, once remarked, "... for fundamental properties we will borrow only names from classical physics." In the present case we would like to borrow from classical mechanics the notion that momentum is the generator of an infinitesimal translation.

$$\mathbf{K} = \frac{\mathbf{p}}{\text{universal constant with the dimension of action}}. \qquad (1.6.30)$$

The universal constant that appears in (1.6.30) turns out to be the same as the constant $\hbar$ that appears in L. de Broglie's relation, written in 1924,

$$\frac{2\pi}{\lambda} = \frac{p}{\hbar}, \qquad (1.6.31)$$

$$\mathscr{T}(d\mathbf{x}') = 1 - i\mathbf{p}\cdot d\mathbf{x}'/\hbar, \qquad (1.6.32)$$

$$[x_i, p_j] = i\hbar\delta_{ij}. \qquad (1.6.33)$$

تمرین

۱– با توجه به خاصیت عملگر انتقال بسیار کوچک، ثابت کنید:

$$\mathscr{T}(d\mathbf{x}'')\mathscr{T}(d\mathbf{x}') = \mathscr{T}(d\mathbf{x}' + d\mathbf{x}'')$$

۲– ثابت کنید:

$$[\mathbf{x}, \mathscr{T}(d\mathbf{x}')]|\mathbf{x}'\rangle = d\mathbf{x}'|\mathbf{x}' + d\mathbf{x}'\rangle \simeq d\mathbf{x}'|\mathbf{x}'\rangle$$

۳– با توجه به رابطه ی قسمت قبل نشان دهید:

$$[x_i, K_j] = i\delta_{ij}$$

چرا در کوانتم p و x جابه جا نمی شوند ولی در کلاسیک می شوند؟

- در کوانتم باید سه تا چیز را با هم تفکیک کنیم:

  ـ حالت سیستم ـ اثر عملگر روی سیستم ـ اندازه گیری

- X و P در کوانتم عملگر هستند! (اندازه گیری باعث تغییر حالت سیستم می شود).
- تکانه مولد انتقال است. یعنی تکانه باعث تغییر مکان می شود.
- **(در نظریه)** XP با PX متفاوت است. چون ترتیب مهم است:

اول انتقال انجام شود، بعد مکان تعیین شود **یا** اول مکان تعیین شود، بعد انتقال صورت گیرد.

- **(در آزمایش/طبیعت):** در کوانتم وقتی صحبت از اندازه گیری می کنیم ما با یک عملیات آماری مواجهیم. یعنی متوسط گیری!

$$\langle(\Delta x)^2\rangle\langle(\Delta p_x)^2\rangle \geq \hbar^2/4$$

$$\left\langle(\Delta x)^2\right\rangle = \langle x^2\rangle - \langle x\rangle^2$$

تابع موج در فضای مختصات

$$x|x'\rangle = x'|x'\rangle \qquad \langle x''|x'\rangle = \delta(x''-x')$$

$$|\alpha\rangle = \int_{-\infty}^{\infty} dx'|x'\rangle\langle x'|\alpha\rangle \qquad |\langle x'|\alpha\rangle|^2 dx' \qquad \int_{-\infty}^{\infty} dx'|\langle x'|\alpha\rangle|^2$$

wave function $\psi_\alpha(x')$ for state $|\alpha\rangle$:

$$\langle x'|\alpha\rangle = \psi_\alpha(x').$$

در مکانیک موجی متداول **ضرایب بسط** و **تابع موج** را در اصول موضوعه ی جداگانه ای معرفی می کنند. ولی در فرمولبندی دیراک این دو به یک اصل موضوعه تبدیل می شوند. چون تابع موج همان ضرایب بسط را به دست می دهد.

$$\langle \beta | \alpha \rangle = \int dx' \langle \beta | x' \rangle \langle x' | \alpha \rangle$$

$$= \int dx' \psi_\beta^*(x') \psi_\alpha(x'), \tag{1.7.6}$$

so $\langle \beta | \alpha \rangle$ characterizes the overlap between the two wave functions. Note that we are not defining $\langle \beta | \alpha \rangle$ as the overlap integral; the identification of $\langle \beta | \alpha \rangle$ with the overlap integral *follows* from our completeness postulate for $|x'\rangle$. The more general interpretation of $\langle \beta | \alpha \rangle$, *independent of representations*, is that it represents the probability amplitude for state $|\alpha\rangle$ to be found in state $|\beta\rangle$.

$$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle$$

$\langle x'|$ on the left

$$\langle x'|\alpha\rangle = \sum_{a'} \langle x'|a'\rangle\langle a'|\alpha\rangle$$

$$\psi_\alpha(x') = \sum_{a'} c_{a'} u_{a'}(x')$$

eigenfunction of operator A with eigenvalue a': $u_{a'}(x') = \langle x'|a'\rangle$

$$\langle\beta|A|\alpha\rangle = \int dx' \int dx'' \langle\beta|x'\rangle\langle x'|A|x''\rangle\langle x''|\alpha\rangle$$
$$= \int dx' \int dx''\, \psi_\beta^*(x')\langle x'|A|x''\rangle\psi_\alpha(x'')$$

$A = x^2$ $\quad \langle x'|x^2|x''\rangle = (\langle x'|)\cdot(x''^2|x''\rangle) = x'^2\delta(x' - x'')$

$$\langle\beta|x^2|\alpha\rangle = \int dx' \langle\beta|x'\rangle x'^2\langle x'|\alpha\rangle = \int dx'\, \psi_\beta^*(x')\, x'^2 \psi_\alpha(x')$$

$$\langle\beta|f(x)|\alpha\rangle = \int dx'\, \psi_\beta^*(x') f(x')\psi_\alpha(x')$$

عملگر تکانه در فضای مختصات

$$\left(1-\frac{ip\Delta x'}{\hbar}\right)|\alpha\rangle = \int dx' \mathscr{T}(\Delta x')|x'\rangle\langle x'|\alpha\rangle$$

$$= \int dx' |x'+\Delta x'\rangle\langle x'|\alpha\rangle$$

$$= \int dx' |x'\rangle\langle x'-\Delta x'|\alpha\rangle$$

$$= \int dx' |x'\rangle\left(\langle x'|\alpha\rangle - \Delta x'\frac{\partial}{\partial x'}\langle x'|\alpha\rangle\right)$$

$$p|\alpha\rangle = \int dx' |x'\rangle\left(-i\hbar\frac{\partial}{\partial x'}\langle x'|\alpha\rangle\right)$$

$$\langle x'|p|\alpha\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|\alpha\rangle$$

$$\langle x'|p|\alpha\rangle = -i\hbar \frac{\partial}{\partial x'} \langle x'|\alpha\rangle$$

$$\langle x'|p|x''\rangle = -i\hbar \frac{\partial}{\partial x'} \delta(x' - x'')$$

$$\langle \beta|p|\alpha\rangle = \int dx' \langle \beta|x'\rangle \left(-i\hbar \frac{\partial}{\partial x'} \langle x'|\alpha\rangle \right)$$

$$= \int dx' \psi_\beta^*(x') \left(-i\hbar \frac{\partial}{\partial x'} \right) \psi_\alpha(x')$$

$$\langle x'|p^n|\alpha\rangle = (-i\hbar)^n \frac{\partial^n}{\partial x'^n} \langle x'|\alpha\rangle$$

$$\langle \beta|p^n|\alpha\rangle = \int dx' \psi_\beta^*(x') (-i\hbar)^n \frac{\partial^n}{\partial x'^n} \psi_\alpha(x')$$

تابع موج در فضای تکانه

$$p|p'\rangle = p'|p'\rangle \qquad |\alpha\rangle = \int dp' |p'\rangle\langle p'|\alpha\rangle$$

momentum-space wave function

$$\langle p'|\alpha\rangle = \phi_\alpha(p')$$

$$\int dp' \langle\alpha|p'\rangle\langle p'|\alpha\rangle = \int dp' |\phi_\alpha(p')|^2 = 1$$

Let us now establish the connection between the x-representation and the p-representation. We recall that in the case of the discrete spectra, the change of basis from the old set $\{|a'\rangle\}$ to the new set $\{|b'\rangle\}$ is characterized by the transformation matrix (1.5.7). Likewise, we expect that the desired information is contained in $\langle x'|p'\rangle$, which is a function of x' and p', usually called the **transformation function** from the x-representation to the p-representation. To derive the explicit form of $\langle x'|p'\rangle$, first recall (1.7.17); letting $|\alpha\rangle$ be the momentum eigenket $|p'\rangle$, we obtain

$$\langle x'|p|p'\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|p'\rangle \tag{1.7.27}$$

or

$$p'\langle x'|p'\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|p'\rangle. \tag{1.7.28}$$

$$p'\langle x'|p'\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|p'\rangle$$

The solution to this differential equation for $\langle x'|p'\rangle$ is

$$\langle x'|p'\rangle = N\exp\left(\frac{ip'x'}{\hbar}\right),$$

$$\langle x'|x''\rangle = \int dp'\,\langle x'|p'\rangle\langle p'|x''\rangle.$$

$$\delta(x'-x'') = |N|^2\int dp'\exp\left[\frac{ip'(x'-x'')}{\hbar}\right]$$

$$= 2\pi\hbar|N|^2\delta(x'-x'').$$

$$\langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ip'x'}{\hbar}\right)$$

$$\langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ip'x'}{\hbar}\right)$$

$$\langle x'|\alpha\rangle = \int dp' \langle x'|p'\rangle\langle p'|\alpha\rangle$$

$$\psi_\alpha(x') = \left[\frac{1}{\sqrt{2\pi\hbar}}\right]\int dp' \exp\left(\frac{ip'x'}{\hbar}\right)\phi_\alpha(p')$$

$$\langle p'|\alpha\rangle = \int dx' \langle p'|x'\rangle\langle x'|\alpha\rangle$$

$$\phi_\alpha(p') = \left[\frac{1}{\sqrt{2\pi\hbar}}\right]\int dx' \exp\left(\frac{-ip'x'}{\hbar}\right)\psi_\alpha(x')$$

تمرین

تابع موج در فضای مکان (x) مربوط به یک بسته موج گاوسی به صورت زیر نشان داده می شود:

$$\langle x'|\alpha\rangle = \left[\frac{1}{\pi^{1/4}\sqrt{d}}\right]\exp\left[ikx' - \frac{x'^2}{2d^2}\right]$$

الف– مقدار چشمداشتی کمیت های x، x^2، p و p^2 را محاسبه کنید.

ب– رابطه ی عدم قطعیت را بررسی کنید: $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle$

پ– رابطه ی مربوط به تابع موج در فضای تکان (p) را به دست آورید.

روابط در ۳ بعد

$$\langle\beta|\mathbf{p}|\alpha\rangle = \int d^3x' \psi_\beta^*(\mathbf{x}')(-i\hbar\nabla)\psi_\alpha(\mathbf{x}')$$

$$\langle\mathbf{x}'|\mathbf{p}'\rangle = \left[\frac{1}{(2\pi\hbar)^{3/2}}\right]\exp\left(\frac{i\mathbf{p}'\cdot\mathbf{x}'}{\hbar}\right)$$

$$\psi_\alpha(\mathbf{x}') = \left[\frac{1}{(2\pi\hbar)^{3/2}}\right]\int d^3p' \exp\left(\frac{i\mathbf{p}'\cdot\mathbf{x}'}{\hbar}\right)\phi_\alpha(\mathbf{p}')$$

$$\phi_\alpha(\mathbf{p}') = \left[\frac{1}{(2\pi\hbar)^{3/2}}\right]\int d^3x' \exp\left(\frac{-i\mathbf{p}'\cdot\mathbf{x}'}{\hbar}\right)\psi_\alpha(\mathbf{x}')$$

It is interesting to check the dimension of the wave functions. In one-dimensional problems the normalization requirement (1.6.8) implies that $|\langle x'|\alpha\rangle|^2$ has the dimension of inverse length, so the wave function itself must have the dimension of $(\text{length})^{-1/2}$. In contrast, the wave function in three-dimensional problems must have the dimension of $(\text{length})^{-3/2}$ because $|\langle\mathbf{x}'|\alpha\rangle|^2$ integrated over all spatial volume must be unity (dimensionless).

تمرین

فضای برداری متشکل از تمام توابع پیوسته $f(x)$ در بازه $[0, 1]$ را در نظر بگیرید. ضرب نرده‌ای در این فضای برداری بصورت $(f, g) = \int_0^1 f(x)g(x)dx$ تعریف شده است. اکنون دنباله‌ای از توابع پیوسته به شکل

$$f_n(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} - \frac{1}{2n} \\ n(x - \frac{1}{2}) + \frac{1}{2}, & \frac{1}{2} - \frac{1}{2n} \leq x \leq \frac{1}{2} + \frac{1}{2n} \\ 1 & \frac{1}{2} + \frac{1}{2n} \leq x \leq 1 \end{cases}$$

در این فضای برداری در نظر بگیرید که $n = 1, 2, \cdots$. نشان دهید این توابع با تعریف کوشی همگرا می‌شوند ولی فضای برداری کامل نیست.

ابتدا هامیلتونی داده شده در مسألهٔ ۱۱ فصل ۱ را به شکل $H = a_0 + \vec{\sigma}.\vec{a}$ بنویسید و سپس با استفاده از ویژه مقادیر و ویژه بردارهای $\vec{S}.\hat{n}$ در مسألهٔ ۹ فصل ۱، ویژه مقادیر و ویژه بردارهای بهنجار هامیلتونی مساله ۱۱ را بدست آورید.

مسائل شماره ۷، ۸، ۲۳ و ۳۳ از فصل اول ساکورای

تمرین

دو عملگر برداری $\vec{A}$ و $\vec{B}$ در نظر بگیرید که مؤلفه‌هایشان در هر دستگاه مختصات دکارتی در روابط جابجایی $[A_i, A_j] = ۰$، $[B_i, B_j] = ۰$ و $[A_i, B_j] = i\hbar\delta_{ij}$ صدق می‌کند که $i, j = ۱, ۲, ۳$. عملگرهایی که نشان دهنده تصویر $\vec{A}$ بر $\vec{B}$ و $\vec{B}$ بر $\vec{A}$ هستند به صورت

$$A_B = \frac{۱}{۲}\left(\vec{A}\cdot\frac{\vec{B}}{B} + \frac{\vec{B}}{B}\cdot\vec{A}\right), \qquad B_A = \frac{۱}{۲}\left(\vec{B}\cdot\frac{\vec{A}}{A} + \frac{\vec{A}}{A}\cdot\vec{B}\right),$$

تعریف می شوند که $A = \sqrt{\vec{A}\cdot\vec{A}}$ و $B = \sqrt{\vec{B}\cdot\vec{B}}$.

آ) نشان دهید

$$\begin{aligned} [A_B, B] &= i\hbar, \\ [B_A, A] &= -i\hbar, \\ \vec{A}\cdot\vec{B} &= AB_A + i\hbar, \\ (\vec{A}\cdot\vec{B})^۲ &= A^۲B_A^۲ + i\hbar\vec{A}\cdot\vec{B}. \end{aligned}$$

ب) برای $\vec{L} = \vec{A}\times\vec{B}$ نشان دهید $L^۲ = B^۲A^۲ - B^۲A_B^۲$.

پ) با انتخاب $\vec{A} = \vec{r}$ و $\vec{B} = \vec{p}$ و استفاده از نتایج فوق، عملگر p_r و r_p را بدست آورید.

ت) شکل هامیلتونی نوسانگر سه‌بعدی $H = \frac{p^۲}{۲m} + \frac{۱}{۲}m\omega^۲r^۲$، را در فضای تکانه بنویسید.

فصل دوم

TIME EVOLUTION AND THE SCHRÖDINGER EQUATION

$$|\alpha, t_0\rangle = |\alpha\rangle \xrightarrow{\text{time evolution}} |\alpha, t_0; t\rangle.$$

time-evolution operator $\mathscr{U}(t, t_0)$:

$$|\alpha, t_0; t\rangle = \mathscr{U}(t, t_0)|\alpha, t_0\rangle.$$

$$|\alpha, t_0\rangle = \sum_{a'} c_{a'}(t_0)|a'\rangle.$$

at some later time, $|\alpha, t_0; t\rangle = \sum_{a'} c_{a'}(t)|a'\rangle.$

$$|c_{a'}(t)| \neq |c_{a'}(t_0)|.$$

$$\sum_{a'} |c_{a'}(t_0)|^2 = \sum_{a'} |c_{a'}(t)|^2$$

fundamental properties of the $\mathscr{U}$ operator

$$\langle \alpha, t_0 | \alpha, t_0 \rangle = 1 \Rightarrow \langle \alpha, t_0; t | \alpha, t_0; t \rangle = 1$$

$$\mathscr{U}^\dagger(t, t_0)\mathscr{U}(t, t_0) = 1$$

$$\mathscr{U}(t_2, t_0) = \mathscr{U}(t_2, t_1)\mathscr{U}(t_1, t_0), \quad (t_2 > t_1 > t_0)$$

$$\lim_{dt \to 0} \mathscr{U}(t_0 + dt, t_0) = 1$$

be unitary: $\mathscr{T}^\dagger(d\mathbf{x}')\mathscr{T}(d\mathbf{x}') = 1.$

$$\mathscr{T}(d\mathbf{x}'')\mathscr{T}(d\mathbf{x}') = \mathscr{T}(d\mathbf{x}' + d\mathbf{x}'').$$

$$\mathscr{T}(-d\mathbf{x}') = \mathscr{T}^{-1}(d\mathbf{x}').$$

$$\lim_{d\mathbf{x}' \to 0} \mathscr{T}(d\mathbf{x}') = 1$$

all these requirements are satisfied by

$$\mathscr{U}(t_0 + dt, t_0) = 1 - i\Omega\, dt \qquad (2.1.15)$$

where Ω is a Hermitian operator,* $\Omega^\dagger = \Omega$

$$\mathscr{U}(t_0 + dt_1 + dt_2, t_0) = \mathscr{U}(t_0 + dt_1 + dt_2, t_0 + dt_1)\,\mathscr{U}(t_0 + dt_1, t_0)$$

$$\mathscr{U}^\dagger(t_0 + dt, t_0)\,\mathscr{U}(t_0 + dt, t_0) = (1 + i\Omega^\dagger\, dt)(1 - i\Omega\, dt) \simeq 1$$

$$\mathscr{T}(d\mathbf{x}') = 1 - i\mathbf{K}\cdot d\mathbf{x}',$$

The operator Ω has the dimension of frequency or inverse time. Is there any familiar observable with the dimension of frequency? We recall that in the old quantum theory, angular frequency ω is postulated to be related to energy by the Planck-Einstein relation

$$E = \hbar\omega. \tag{2.1.19}$$

Let us now borrow from classical mechanics the idea that the Hamiltonian is the generator of time evolution (Goldstein 1980, 407–8). It is then natural to relate Ω to the Hamiltonian operator H:

$$\Omega = \frac{H}{\hbar}. \tag{2.1.20}$$

$$\mathscr{U}(t_0 + dt, t_0) = 1 - \frac{iH\,dt}{\hbar}, \tag{2.1.21}$$

کمیت توصیف کننده حالت سیستم	مولد تغییر
مختصات، مکان یک ذره x	تکانه P
زمان t	انرژی، هامیلتونی H
؟؟؟	چرخش S، L

The Schrödinger Equation

$$\mathscr{U}(t+dt, t_0) - \mathscr{U}(t, t_0) = -i\left(\frac{H}{\hbar}\right) dt\, \mathscr{U}(t, t_0), \qquad (2.1.24)$$

Schrödinger equation for the time-evolution operator

$$i\hbar \frac{\partial}{\partial t} \mathscr{U}(t, t_0) = H \mathscr{U}(t, t_0). \qquad (2.1.25)$$

$$\longrightarrow \quad i\hbar \frac{\partial}{\partial t} \mathscr{U}(t, t_0)|\alpha, t_0\rangle = H \mathscr{U}(t, t_0)|\alpha, t_0\rangle.$$

Schrödinger equation for the state ket

$$i\hbar \frac{\partial}{\partial t} |\alpha, t_0; t\rangle = H|\alpha, t_0; t\rangle, \qquad (2.1.27)$$

If we are given $\mathcal{U}(t, t_0)$ and, in addition, know how $\mathcal{U}(t, t_0)$ acts on the initial state ket $|\alpha, t_0\rangle$, it is not necessary to bother with the Schrödinger equation for the state ket (2.1.27). All we have to do is apply $\mathcal{U}(t, t_0)$ to $|\alpha, t_0\rangle$; in this manner we can obtain a state ket at any t.

$$i\hbar \frac{\partial}{\partial t} \mathcal{U}(t, t_0) = H \mathcal{U}(t, t_0)$$

Case 1. The Hamiltonian operator is independent of time.

$$\mathcal{U}(t, t_0) = \exp\left[\frac{-iH(t-t_0)}{\hbar}\right]. \tag{2.1.28}$$

To prove this let us expand the exponential as follows:

$$\exp\left[\frac{-iH(t-t_0)}{\hbar}\right] = 1 - \frac{iH(t-t_0)}{\hbar} + \left[\frac{(-i)^2}{2}\right]\left[\frac{H(t-t_0)}{\hbar}\right]^2 + \cdots. \tag{2.1.29}$$

Because the time derivative of this expansion is given by

$$\frac{\partial}{\partial t}\exp\left[\frac{-iH(t-t_0)}{\hbar}\right] = \frac{-iH}{\hbar} + \left[\frac{(-i)^2}{2}\right] 2\left(\frac{H}{\hbar}\right)^2 (t-t_0) + \cdots, \tag{2.1.30}$$

$$\lim_{N\to\infty}\left[1 - \frac{(iH/\hbar)(t-t_0)}{N}\right]^N = \exp\left[\frac{-iH(t-t_0)}{\hbar}\right]. \tag{2.1.31}$$

Case 2. The Hamiltonian operator H is time-dependent but the H's at different times commute.

$$\mathscr{U}(t, t_0) = \exp\left[-\left(\frac{i}{\hbar}\right)\int_{t_0}^{t} dt' H(t')\right]. \tag{2.1.32}$$

Case 3. The H's at different times do *not* commute.

$$\mathscr{U}(t, t_0) = 1 + \sum_{n=1}^{\infty}\left(\frac{-i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n H(t_1) H(t_2) \cdots H(t_n), \tag{2.1.33}$$

which is sometimes known as the **Dyson series**, after F. J. Dyson, who developed a perturbation expansion of this form in quantum field theory.

Energy Eigenkets

To be able to evaluate the effect of the time-evolution operator (2.1.28) on a general initial ket $|\alpha\rangle$, we must first know how it acts on the base kets used in expanding $|\alpha\rangle$. This is particularly straightforward if the base kets used are eigenkets of A such that

$$[A, H] = 0; \tag{2.1.34}$$

then the eigenkets of A are also eigenkets of H, called **energy eigenkets**, whose eigenvalues are denoted by $E_{a'}$:

$$H|a'\rangle = E_{a'}|a'\rangle. \tag{2.1.35}$$

We can now expand the time-evolution operator in terms of $|a'\rangle\langle a'|$. Taking $t_0 = 0$ for simplicity, we obtain

$$\exp\left(\frac{-iHt}{\hbar}\right) = \sum_{a'}\sum_{a''}|a''\rangle\langle a''|\exp\left(\frac{-iHt}{\hbar}\right)|a'\rangle\langle a'| = \sum_{a'}|a'\rangle\exp\left(\frac{-iE_{a'}t}{\hbar}\right)\langle a'|. \tag{2.1.36}$$

$$\boxed{\exp\left(\frac{-iHt}{\hbar}\right) = \sum_{a'}|a'\rangle\exp\left(\frac{-iE_{a'}t}{\hbar}\right)\langle a'|.}$$

As an example,

$$|\alpha\rangle = |\alpha, t_0 = 0\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle = \sum_{a'} c_{a'}|a'\rangle.$$

$$|\alpha, t\rangle = |\alpha, t_0 = 0; t\rangle = \exp\left(\frac{-iHt}{\hbar}\right)|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle\exp\left(\frac{-iE_{a'}t}{\hbar}\right).$$

$$\longrightarrow \quad c_{a'}(t=0) \rightarrow c_{a'}(t) = c_{a'}(t=0)\exp\left(\frac{-iE_{a'}t}{\hbar}\right)$$

$$|\alpha\rangle = |a'\rangle \quad \longrightarrow \quad |\alpha, t\rangle = |a'\rangle\exp\left(\frac{-iE_{a'}t}{\hbar}\right),$$

so if the system is initially a simultaneous eigenstate of A and H, it remains so at all times. The most that can happen is the phase modulation, $\exp(-iE_{a'}t/\hbar)$. It is in this sense that an observable compatible with H [see (2.1.34)] is a *constant of the motion*.

In the foregoing discussion the basic task in quantum dynamics is reduced to finding an observable that commutes with H and evaluating its eigenvalues. Once that is done, we expand the initial ket in terms of the eigenkets of that observable and just apply the time-evolution operator. This last step merely amounts to changing the phase of each expansion coefficient, as indicated by (2.1.39).

Even though we worked out the case where there is just one observable A that commutes with H, our considerations can easily be generalized when there are several mutually compatible observables all also commuting with H:

$$[A,B]=[B,C]=[A,C]=\cdots=0,$$
$$[A,H]=[B,H]=[C,H]=\cdots=0. \tag{2.1.42}$$

Using the collective index notation of Section 1.4 [see (1.4.37)], we have

$$\exp\left(\frac{-iHt}{\hbar}\right)=\sum_{K'}|K'\rangle\exp\left(\frac{-iE_{K'}t}{\hbar}\right)\langle K'|, \tag{2.1.43}$$

where $E_{K'}$ is uniquely specified once $a', b', c', \ldots$ are specified. It is therefore of fundamental importance to find *a complete set of mutually compatible observables that also commute with H*. Once such a set is found, we express the initial ket as a superposition of the simultaneous eigenkets of $A, B, C, \ldots$ and H. The final step is just to apply the time-evolution operator, written as (2.1.43). In this manner we can solve the most general initial-value problem with a time-independent H.

$$[\mathbf{x}, \mathscr{T}(d\mathbf{x}')] = d\mathbf{x}', \tag{1.6.25}$$

$$[x_i, K_j] = i\delta_{ij}, \tag{1.6.27}$$

J. Schwinger, lecturing on quantum mechanics, once remarked, "… for fundamental properties we will borrow only names from classical physics." In the present case we would like to borrow from classical mechanics the notion that momentum is the generator of an infinitesimal translation. An infinitesimal translation in classical mechanics can be regarded as a canonical transformation,

$$\mathbf{x}_{\text{new}} \equiv \mathbf{X} = \mathbf{x} + d\mathbf{x}, \qquad \mathbf{p}_{\text{new}} \equiv \mathbf{P} = \mathbf{p}, \tag{1.6.28}$$

obtainable from the generating function (Goldstein 1980, 395 and 411)

$$F(\mathbf{x}, \mathbf{P}) = \mathbf{x} \cdot \mathbf{P} + \mathbf{p} \cdot d\mathbf{x}, \tag{1.6.29}$$

(1.6.29) is the generating function for the identity transformation ($\mathbf{X} = \mathbf{x}, \mathbf{P} = \mathbf{p}$). We are therefore led to speculate that the operator $\mathbf{K}$ is in some sense related to the momentum operator in quantum mechanics.